Prove that $\sum_{i=2}^{n}(2i-3)^2 = \frac{(n-1)(2n-1)(2n-3)}{3}$ for all integers $n \geq 2$ **using mathematical induction**. **SCORE: _____ / 15 PTS**

GRADED BY ME

BASIS STEP: $\sum_{i=2}^{2}(2i-3)^2 = 1^2 = 1$

$\frac{1(3)(1)}{3} = 1$ ✓

INDUCTIVE STEP: ASSUME $\sum_{i=2}^{k}(2i-3)^2 = \frac{(k-1)(2k-1)(2k-3)}{3}$ FOR SOME ARBITRARY INTEGER $k \geq 2$

PROVE $\sum_{i=2}^{k+1}(2i-3)^2 = \frac{k(2k+1)(2k-1)}{3}$

$$\sum_{i=2}^{k+1}(2i-3)^2$$

$$= \sum_{i=2}^{k}(2i-3)^2 + (2(k+1)-3)^2$$

$$= \sum_{i=2}^{k}(2i-3)^2 + (2k-1)^2$$

$$= \frac{(k-1)(2k-1)(2k-3)}{3} + (2k-1)^2$$

$$= \frac{(k-1)(2k-1)(2k-3) + 3(2k-1)^2}{3}$$

$$= \frac{2k-1}{3}\left[(k-1)(2k-3) + 3(2k-1)\right]$$

$$= \frac{2k-1}{3}(2k^2-5k+3+6k-3)$$

$$= \frac{2k-1}{3}(2k^2+k)$$

$$= \frac{k(2k+1)(2k-1)}{3}$$

SO, BY MI,

$$\sum_{i=2}^{n}(2i-3)^2 = \frac{(n-1)(2n-1)(2n-3)}{3}$$

FOR ALL INTEGERS $n \geq 2$

Expand and completely simplify $(2t^3 - \frac{1}{\sqrt{t}})^5$.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

SCORE: _____ / 7 PTS

$$(2t^3)^5 \underbrace{- 5(2t^3)^4(t^{-\frac{1}{2}})}_{1} \underbrace{+ 10(2t^3)^3(t^{-\frac{1}{2}})^2}_{1} \underbrace{- 10(2t^3)^2(t^{-\frac{1}{2}})^3}_{1} \underbrace{+ 5(2t^3)(t^{-\frac{1}{2}})^4}_{1} - (t^{-\frac{1}{2}})^5$$

$$= \underbrace{32t^{15}}_{\frac{1}{2}} \underbrace{- 80t^{\frac{23}{2}}}_{\frac{1}{2}} \underbrace{+ 80t^{8}}_{\frac{1}{2}} \underbrace{- 40t^{\frac{9}{2}}}_{\frac{1}{2}} \underbrace{+ 10t}_{\frac{1}{2}} \underbrace{- t^{-\frac{5}{2}}}_{\frac{1}{2}}$$

Consider the expansion of $(3x^2 + 5y^3)^{24}$.

SCORE: _____ / 8 PTS

For the questions below, you may write the coefficients in your final answers in factored form, as shown in lecture.
The coefficients in your final answers must **NOT** contain division, ! nor $C(n, r)$ (or equivalent) notation.

[a] Find the first 3 terms in the expansion.

$$\binom{24}{0}(3x^2)^{24} + \binom{24}{1}(3x^2)^{23}(5y^3) + \binom{24}{2}(3x^2)^{22}(5y^3)^2$$

$$= \underbrace{3^{24}x^{48}}_{\frac{1}{2}} + \underbrace{24 \cdot 3^{23} \cdot 5 x^{46} y^3}_{1} + \underbrace{12 \cdot 23 \cdot 3^{22} \cdot 5^2 x^{44} y^6}_{1} \quad \frac{1}{2} \quad \frac{1}{2}$$

$$\binom{24}{0} = \frac{24!}{0!\,24!} = 1$$

$$\binom{24}{1} = \frac{24!}{1!\,23!} = \frac{24 \cdot 23!}{1 \cdot 23!} = 24$$

$$\binom{24}{2} = \frac{24!}{2!\,22!} = \frac{\overset{12}{24} \cdot 23 \cdot 22!}{2 \cdot 1 \cdot 22!}$$

$$\binom{24}{18} = \frac{24!}{18!\,6!} = \frac{24 \cdot 23 \cdot 22 \cdot \overset{7}{21} \cdot \overset{2}{20} \cdot 19 \cdot 18!}{18! \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

[b] Find the coefficient of $x^{12}y^{54}$ in the expansion.

$$\sum_{i=0}^{24} \binom{24}{i}(3x^2)^{24-i}(5y^3)^i = \sum_{i=0}^{24} \binom{24}{i} 3^{24-i} 5^i x^{48-2i} y^{3i} \longrightarrow 3i = 54 \longrightarrow i = 18$$

$$1 \quad 2\tfrac{1}{2}$$

$$1 \quad \underbrace{\binom{24}{18}}\underbrace{3^6 5^{18} x^{12} y^{54}} = \underbrace{23 \cdot 22 \cdot 7 \cdot 2 \cdot 19 \cdot 3^6 \cdot 5^{18} x^{12} y^{54}}$$